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Regularized Solutions for Abstract Volterra Equations

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We introduce a two-kernel dependent family of strong continuous operators defined in a Banach space, which allows us to consider in an unified treatment the notions of, among others, C_0-semigroups of operators, cosine families, n-times integrated semigroups, resolvent families and k-generalized solutions.

The results are applied to the study of existence and uniqueness of solutions for the Volterra equation of convolution type $u = f + a * Au$, in the case A is not necessarily densely defined. Examples for equations defined in L^p spaces are also given.

Key Words: abstract linear Volterra equations; convolution; k-regularized resolvents; semigroups of linear operators.

1. INTRODUCTION

We consider the Volterra equation of scalar type defined on a complex Banach space X

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \qquad t \in J, \tag{1.1}$$

where A is a closed linear unbounded operator with domain $D(A)$, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ is a scalar kernel $\neq 0$, and $f \in C(J, X)$, $J := [0, T]$.

A theory for the abstract Volterra equation (1.1) has been developed due to its many applications to problems in physics, engineering, and biology; see [4, 12–14, 17, 19] and the monograph of Prüss [18] and the references therein. In all of these papers, the density of the domain of A is required, but there are cases where such a condition is not satisfied.

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The basic concept concerning (1.1) is that of well-posedness which is the direct extension of the corresponding notion usually employed for the abstract Cauchy problem of first order

$$\dot{u}(t) = Au(t), \qquad u(0) = u_0. \tag{1.2}$$

It is well known that well-posedness is equivalent to the existence of a resolvent $\{S(t)\}_{t \geq 0} \subset \mathscr{B}(X)$ for (1.1), i.e., a strongly continuous family of bounded linear operators in X which commutes with A and satisfies the resolvent equation

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds, \qquad t \geq 0, x \in D(A). \tag{1.3}$$

The resolvent is the central object to be studied in the theory of Volterra equations; it corresponds to the strongly continuous semigroup generated by A in the special case $a(t) \equiv 1$ and $f(t) \equiv u_0$, i.e., for (1.2). The importance of the resolvent $S(t)$ is shown by the variation of parameters formula

$$u(t) = S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)\,ds, \qquad t \in J, \tag{1.4}$$

where $f \in W^{1,1}(J; X)$.

In this paper we will be concerned with the study of existence, uniqueness, and some qualitative properties of solutions for the abstract Volterra equation (1.1) by means of an extended notion of resolvent. We will see that this new concept is very natural when we handle the equation (1.1) with A the generator of an n-times integrated semigroup [10]. Some of the immediate advantages of our approach are the treatment of (1.1) in cases where the domain of A is not necessarily dense in X and the analysis realized in the *local* context. On the other hand, the framework presented in this paper seems to be very useful to extend and improve most of those known results for integrated resolvents, integrated semigroups, and integrated cosine functions.

The basic idea is to regularize the Eq. (1.1) by introducing an arbitrary scalar kernel in (1.3) (see Definition 2.1 below). This method has been recently applied successfully to the abstract Cauchy problem (1.2) by Cioranescu and Lumer [5] and Bäumer and Neubrander [3]. Roughly speaking, the idea is the following: We write (1.1) as $u = f + A(a * u)$ and convoluting with a scalar kernel $k \neq 0$ we obtain $k * u = k * f + A(a * (k * u))$. This shows that if u is a solution of (1.1) then $v = k * u$ is a "k-regularized" solution of the convoluted equation. Conversely, let v be a k-regularized solution of the equation $v = k * f + A(a * v)$ and assume that there is u such that $v = k * u$. Then, by Titschmarsch's theorem, we conclude that u is a solution of (1.1). We observe that if we follow this procedure for $k(t) = t^n/n!$ one obtains the method of n-times integrated resolvent families studied in detail in [11, 16] and introduced in [2].

In the next section we define the notion of k-regularized resolvent and given some properties. We show the importance of the kernel $k(t)$ in its role as "smoothing operator" and that the local behavior $(t \to 0)$ of $k(t)$ is very important for this purpose. Assuming existence of a k-regularized resolvent for (1.1), the main result in Section 2 (Theorem 2.7) characterizes existence and uniqueness of solutions for (1.1) by means of a simple condition involving the range of the convolution transform $Ku = k * u$ and the forcing function f. Then, some applications are derived, extending in particular results in [2, 7, 11, 16].

In Section 3, we first study the existence of k-regularized resolvents under conditions of Hille–Yosida type. Next, we find conditions on $a(t)$ for existence of k-regularized resolvents for (1.1) in the case A is the generator of an n-times integrated semigroup (Theorem 3.7). In particular, we recover and extend results of [16, 17].

Throughout this paper we follow the notation used in the book [18].

2. k-REGULARIZED RESOLVENTS

In 1987, Arendt [1] introduced the notion of n-times integrated semigroups to treat (1.2) in some cases where the Hille–Yosida theorem does not apply. Later, Hieber [7] extended the definition to α-times integrated semigroups, $\alpha \in \mathbb{R}_+ = [0, \infty)$ and applied them to the study of differential operators in L^p spaces (see also [8, 9]).

For the case of the abstract Cauchy problem of second order, an analogous theory has been developed (see, e.g., El-Mennaoui and Keyantuo [6] and the references therein).

Concerning Eq. (1.1), Arendt and Kellermann [2] considered the notion of n-times integrated resolvent families (see also Kim [11] and Oka [15, 16] for recent results).

DEFINITION 2.1. Let $k \in C(\mathbb{R}_+)$ be a scalar kernel. A family $\{R(t)\}_{t \geq 0} \subset \mathscr{B}(X)$ is called a k-regularized resolvent for (1.1) if the following conditions are satisfied:

(R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = k(0)I$;

(R2) $R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(R3) the k-regularized resolvent equation holds

$$R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x\,ds \tag{2.1}$$

for all $x \in D(A)$, $t \geq 0$.

For example, if we take $k(t) = t^n/n!$ in Definition 2.1, we obtain that a k-generalized resolvent is precisely an n-times integrated resolvent (see [2, Definition 1.1]); in particular, it contains the notions of C_0-semigroups (with $n = 0$ and $a(t) \equiv 1$); cosine families (with $n = 0$ and $a(t) \equiv t$); n-times integrated semigroups (with $a(t) \equiv 1$) and n-times integrated cosine functions (with $a(t) \equiv t$). On the other hand, if we put $a(t) \equiv 1$ we obtain the notion of k-convoluted semigroups or k-regularized solutions for the abstract Cauchy problem of first order, introduced by Cioranescu and Lumer [5] and Bäumer and Neubrander [3]. Finally, taking $k(t) = a(t)$ we obtain the concept of integral resolvents [18], but in this case $a(t)$ must be continuous.

We remark that it is possible to take another way to define the notion of k-regularized resolvents. In fact, we can follow for example Hieber [7] or Arendt and Kellermann [2] and, by making use of vector-valued Laplace transform theory, we obtain an equivalent definition (see Section 3, Proposition 3.1). However, this approach loses the local character of the Eq. (1.1).

In what follows we study the relation between k-regularized resolvent families and resolvent families. Throughout this section we assume $\rho(A) \neq \emptyset$.

We start with the following simple but important lemma.

LEMMA 2.2. *Suppose* (1.1) *admits a k-regularized resolvent. Then* $a * R(t)x \in D(A)$ *for all* $x \in X$, $t \geq 0$ *and*

$$R(t)x = k(t)x + A\int_0^t a(t-s)R(s)x\,ds, \qquad x \in X, t \geq 0. \tag{2.2}$$

Proof. Let $x \in X$ and define $y = (\lambda - A)^{-1}x \in D(A)$, where $\lambda \in \rho(A)$ is fixed. Let $z = a * R(t)x$, $t \geq 0$. We have to show that $z \in D(A)$ and $Az = R(t)x - k(t)x$. Using (R2) and (R3) we obtain $z = (\lambda - A)a * R(t)y = \lambda a * R(t)y - a * AR(t)y = \lambda a * R(t)y - (R(t)y - k(t)y) \in D(A)$ and $(\lambda - A)z = \lambda a * R(t)x - (R(t)x - k(t)x) = \lambda z - (R(t)x - k(t)x)$, which gives (2.2). ∎

COROLLARY 2.3. *If $R_1(t)$ is a k_1-regularized resolvent and $R_2(t)$ is a k_2-regularized resolvent for* (1.1), *then* $(k_1 * R_2)(t) = (k_2 * R_1)(t)$ *for all* $t \geq 0$.

Proof. For $x \in D(A)$ we obtain $k_2 * R_1 x = (R_2 - a * AR_2) * R_1 x = R_2 * R_1 - a * R_2 * AR_1 x = R_2 * (R_1 x - a * AR_1 x) = R_2 * k_1 x = k_1 * R_2 x$. Hence $(k_2 * R_1)(t)x = (k_1 * R_2)(t)x$ for each $x \in D(A)$, $t \geq 0$. Let $\lambda \in \rho(A)$ and $y \in X$. Define $x = (\lambda - A)^{-1} y$. Then $(\lambda - A)(k_2 * R_1)(t)x = (\lambda - A)(k_1 * R_2)(t)x$ implies $(k_2 * R_1)(t)y = (k_1 * R_2)(t)y$ for each $y \in X$, $t \geq 0$. ∎

Remarks 2.4. (1) As a consequence of Corollary 2.3, and Titchmarsch's Theorem, (1.1) admits at most one k-generalized resolvent $R(t)$.

(2) If $R(t)$ is a k_1-regularized resolvent and a k_2-regularized resolvent, then $k_1 = k_2$.

(3) Let $R_i(t)$ be k_i-regularized resolvents for (1.1) $(i = 1, 2)$. Then $(R_1 + R_2)(t)$ is a $(k_1 + k_2)$-regularized resolvent.

(4) If $R(t)$ is a k-regularized resolvent for (1.1), $k \in AC_{\text{loc}}(\mathbb{R}_+)$ and $b \in L^1_{\text{loc}}(\mathbb{R}_+)$, then $(b * R)(t)$ is a $(b * k)$-regularized resolvent for (1.1). In particular, suppose $S(t)$ is a resolvent for (1.1). Then (1.1) admits a k-regularized resolvent $R(t)$ given by

$$R(t)x = \frac{d}{dt}\left(\int_0^t k(t-s)S(s)x\,ds\right) \tag{2.3}$$

for all $x \in D(A)$ and $t \in J$.

Conversely, we have the following useful criterion for applications.

PROPOSITION 2.5. *Assume* (1.1) *admits a k-regularized resolvent with $k \in AC_{\text{loc}}(\mathbb{R}_+)$ and $k(0) \neq 0$. Then* (1.1) *admits a resolvent.*

Proof. Let $b \in L^1_{\text{loc}}(\mathbb{R}_+)$ be such that $b * \dot{k} = -\dot{k}k(0)^{-1} - bk(0)$. Convoluting with $1(t) \equiv 1$, we obtain

$$(b * k)(t) + k(t)k(0)^{-1} = 1 \qquad \text{for all } t \geq 0. \tag{2.4}$$

We claim that (1.1) admits a resolvent $S(t)$ given by

$$S(t)x = k(0)^{-1}R(t)x + \int_0^t b(t-s)R(s)x\,ds \tag{2.5}$$

for all $x \in D(A)$ and $t \geq 0$.

In fact, it is clear that $S(t)$ is strongly continuous on $\mathbb{R}_+$, $S(0) = I$, and $S(t)$ commutes with A for each $t \geq 0$. For all $x \in D(A)$ we obtain by (2.1) and (2.4)

$$\begin{aligned} Sx &= k(0)^{-1} Rx + b * Rx = k(0)^{-1}(kx + a * ARx) + b * (kx + a * ARx) \\ &= k(0)^{-1} kx + k(0)^{-1} a * ARx + b * kx + b * (a * AR) x \\ &= x + k(0)^{-1} a * ARx + b * (a * AR) x \\ &= x + a * A\big(k(0)^{-1} Rx + b * Rx\big) = x + a * ASx. \end{aligned}$$

Thus $S(t)x = x + (a * AS)(t)x$ for each $x \in D(A)$ and $t \geq 0$, which proves the claim. ∎

Remarks 2.6. (1) From the above proof we note that we only need the condition (2.4) to get existence of a resolvent.

(2) Suppose $f \in W^{1,1}_{\mathrm{loc}}(J, X)$ and $k \in AC_{\mathrm{loc}}(\mathbb{R}_+)$, $k(0) \neq 0$. For the special case $a(t) \equiv 1$, (1.1) is equivalent to

$$\dot{u}(t) = \dot{f}(t) + Au(t), \qquad u(0) = f(0). \tag{2.6}$$

Proposition 2.5 states that (2.6) admits a k-convoluted semigroup [5] if and only if A generates a C_0-semigroup in X.

(3) Examples of k-regularized resolvents which cannot be resolvents are given by the kernels $k(t) = t^{\alpha}/\Gamma(\alpha + 1)$, $\alpha > 0$. By $\alpha = n$ a positive integer, they are the n-times integrated resolvents treated in [11, 16] (see also the references therein). Note that in this case $k(0) = 0$.

In what follows, we assume existence of a k-regularized resolvent (which will be treated in the next section) and study existence and uniqueness of solutions for (1.1).

We recall that a function $u \in C(J, X)$ is called a *solution* of (1.1) on J if $a * u \in C(J, [D(A)])$ and $u(t) = f(t) + A(a * u)(t)$ on J; it is called a *weak solution* of (1.1) on J if $\langle u(t), x^* \rangle = \langle f(t), x^* \rangle + \langle (a * u)(t), y^* \rangle$ on J, for each $x^* \in D(A^*)$ and $y^* \in A^* x^*$. Observe that A^* is multivalued unless A is densely defined.

Let $k \in C(\mathbb{R}_+)$ be given and denote by K: $C(J, X) \to C(J, X)$ the *convolution transform* defined as $Ku = k * u$. Note that K is linear, bounded, and, by Titchmarsch's theorem, an injective operator.

The next theorem is the main result of this section.

THEOREM 2.7. *Let $f \in C(J, X)$ and suppose* (1.1) *admits a k-regularized resolvent $R(t)$. Then there is a unique solution of* (1.1) *if and only if* $R * f \in \mathrm{Ran}(K)$.

Proof. Let $u(t)$ be the unique solution of (1.1). Then from (R1)–(R3) and Lemma 2.2 we obtain

$$\begin{aligned} R*f &= R*(u - A(a*u)) = R*u - R*A(a*u) \\ &= R*u - AR*(a*u) = (R - A(a*R))*u = k*u. \end{aligned}$$

Conversely, suppose $R*f$ is in the range of the convolution transform. Then there is a unique $u \in C(J, X)$ such that $k*u = R*f$. Thus, we get from Lemma 2.2

$$(a*u)*k = a*(R*f) = (a*R)*f \in D(A),$$

for each $t \in J$, and

$$\begin{aligned} A((a*u)*k) &= A(a*R)*f = (R-k)*f = R*f - k*f \\ &= k*u - k*f. \end{aligned}$$

Hence, for each $x^* \in D(A^*)$ and $y^* \in A^*x^*$,

$$\begin{aligned} k*\langle u, x^*\rangle = \langle k*u, x^*\rangle &= \langle k*f, x^*\rangle + \langle a*u*k, y^*\rangle \\ &= k*\langle f, x^*\rangle + k*\langle a*u, y^*\rangle. \end{aligned}$$

By Titchmarsch's theorem we get

$$\langle u, x^*\rangle = \langle f, x^*\rangle + \langle a*u, y^*\rangle.$$

Hence $u \in C(J, X)$ is a weak solution of (1.1) on J.

Let $\lambda \in \rho(A)$. Then for each $x^* \in D(A^*)$ and $y^* \in A^*x^*$

$$\langle a*u, \lambda x^* - y^*\rangle = \langle \lambda a*u - u + f, x^*\rangle.$$

Let $z^* \in X^*$. Since $\lambda \in \rho(A^*)$, there exists an $x_0^* \in D(A^*)$ such that $z^* \in (\lambda - A^*)x_0^*$. It follows that $z^* = \lambda x_0^* - y_0^*$ for some $y_0^* \in A^*x_0^*$. Hence $(\lambda - A^*)^{-1}z^* = ((\lambda - A)^{-1})^* z^* = x_0^*$ and

$$\langle a*u, z^*\rangle = \left\langle \lambda a*u - u + f, \left((\lambda - A)^{-1}\right)^* z^*\right\rangle.$$

Thus,

$$\langle a*u, z^*\rangle = \left\langle (\lambda - A)^{-1}(\lambda a*u - u + f), z^*\right\rangle.$$

Since $z^* \in X^*$ is arbitrary, it follows that $(a*u)(t) = (\lambda - A)^{-1}(\lambda(a*u)(t) - u(t) + f(t))$ for all $t \in J$. Therefore, $\lambda(a*u)(t) - A(a*u)(t) = \lambda(a*u)(t) - u(t) + f(t)$. From this equation we conclude that u is the unique solution of (1.1). ∎

Remarks 2.8.

(1) In many cases, $\operatorname{Ran}(K)$ is the domain of a differential operator; see [20].

(2) For $k = 1, 2, \ldots$ we denote by $C^{n,k}(J, X)$ the set of all n-times continuously differentiable functions $v\colon J \to X$ such that $v(0) = v'(0) = \cdots = v^{(k-1)}(0) = 0$. If $k = 0$ or $k = n$, we denote $C^{n,0}(J, X) \equiv C^{(n)}(J, X)$ and $C_0^n(J, X) \equiv C^{n,n}(J, X)$, respectively. With the above notation, we can check that $\operatorname{Ran}(K) \subset C^{n,k+1}(J; X)$ for all $n, k \in \mathbb{N}_0$, whenever $k \in C^{n,k}(J; \mathbb{C})$.

(3) Theorem 2.7 says that $u \in C(J, X)$ is a solution of (1.1) if and only if the following "generalized variation of parameters formula" holds:

$$\int_0^t k(t-s)u(s)\,ds = \int_0^t R(t-s)f(s)\,ds. \tag{2.7}$$

For, if $f \in W^{1,1}(J, X)$ and $k(t) \equiv 1$ then, after differentiation, (2.7) coincides with (1.4).

By means of Theorem 2.7 we can obtain results on existence and uniqueness of solutions for (1.1).

COROLLARY 2.9. *Suppose* (1.1) *admits an n-times integrated resolvent $R(t)$. Let $f \in C(J, X)$. Then there is a unique solution for* (1.1) *if and only if $R * f \in C_0^{n+1}(J, X)$.*

Proof. Suppose that (1.1) admits an n-times integrated resolvent and define $B\colon C_0^{n+1}(J, X) \to C(J, X)$ as $Bv = v^{(n+1)}$. Integration by parts gives

$$KBv(t) = \int_0^t \frac{(t-s)^n}{n!} v^{(n+1)}(s)\,ds = v(t).$$

This proves that $C_0^{n+1}(J, X) \subseteq \operatorname{Ran}(K)$. Conversely, note that $k(t) = t^n/n!$ is such that $k \in C^{p,n}(J; \mathbb{C})$ for all $p \in \mathbb{N}$. Thus, $\operatorname{Ran}(K) = C_0^{n+1}(J, X)$. ∎

The above corollary, together with the following remark, recovers Arendt and Kellermann's result [2, Theorem 1.2].

Remark 2.10. Observe that the property $k \in C_0^n(J; \mathbb{C})$ implies $R * f \in C_0^{n+1}(J, X)$ if and only if $R * f \in C^{(n+1)}(J, X)$. In fact, assume that $w := R * f \in C^{(n+1)}(J, X)$. Then by (R3) we have $R * f = (k + A(a * R)) * f = k * f + A(a * R * f)$. Hence

$$w = k * f + A(a * w).$$

Since A is closed and $w(0) = 0$, we can differentiate this to obtain

$$w' = k' * f + A(a * w')$$

and so $w'(0) = 0$. After differentiation of this equation, we have $w^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots, n$, i.e., $R * f \in C_0^{n+1}(J, X)$.

Combining this remark with Corollary 2.9, we obtain the following result.

COROLLARY 2.11. *Suppose* (1.1) *admits an n-times integrated resolvent $R(t)$. Assume $a \in BV_{\text{loc}}(\mathbb{R}_+)$ if $n \geq 1$. Let $f \in C^{(n+1)}(J, X)$ such that $f^{(k-1)}(0) \in D(A^{n+1-k})$ $(k = 1, 2, \ldots, n+1)$. Then* (1.1) *has a unique solution.*

Proof. We claim that $(R * f) \in C^{(n+1)}(J, X)$. In fact, we show by induction that $(R * f)^{(m)} \in C^{(m)}(J, X)$ and

$$\begin{aligned}(R * f)^{(m)}(t) &= R^{(m-1)}f(0) + R^{(m-2)}f'(0) + \cdots + R^{(m-k)}f^{(k-1)}(0) + \cdots \\ &\quad + Rf^{(m-1)}(0) + R * f^{(m)}(t) \qquad (m = 1, 2, \ldots, n+1).\end{aligned} \tag{2.8}$$

For $m = 1$ this is proved by differentiating $R * f$ by the usual rule. Assume that (2.8) holds for $m \leq n + 1$. Note that $R(t)y \in C^{(m)}(J, X)$ if $y \in D(A^m)$, because of (R3) and Lemma 2.2 (see also [16, p. 282]). Hence, $(R * f)^{(m)} \in C(J, X)$ and we obtain (2.8) by differentiating. We have shown that $(R * f) \in C^{(n+1)}(J, X)$, which implies the claim. ∎

EXAMPLE 2.12. As an application of Theorem 2.7, consider a k-regularized resolvent $R(t)$ and suppose $k \in C^1(J)$ with $k(0) \neq 0$. Define B: $C_0^1(J, X) \to C(J, X)$ as $Bv(t) = v'(t)k(0)^{-1} + (b * v')(t)$. We have $KBv(t) = (k * v')(t)k(0)^{-1} + ((k * b) * v')(t) = (k * v')(t)k(0)^{-1} + (1 * v')(t) - (k * v')(t)k(0)^{-1} = v(t)$. Since $k \in C^{1,0}(J, \mathbb{C})$ we obtain by Remark 2.8 that $\text{Ran}(K) = C_0^1(J, X)$. Combining with Remark 2.10 we obtain the following result.

COROLLARY 2.13. *Suppose* (1.1) *admits a k-regularized resolvent with $k \in AC_{\text{loc}}(J)$ and $k(0) \neq 0$. Assume $f \in C^1(J, X)$. Then there exists a unique solution for* (1.1).

3. EXISTENCE OF k-REGULARIZED RESOLVENTS

We assume throughout this section that $a, k \in L^1_{\text{loc}}(\mathbb{R}_+)$ are Laplace transformable and there is an $\omega \in \mathbb{R}$ such that $\hat{a}(\lambda) \neq 0$ for all $\lambda > \omega$.

PROPOSITION 3.1. *Let $R(t)$ be an exponentially bounded and strongly continuous operator family in $\mathscr{B}(X)$ such that the Laplace transform $\hat{R}(\lambda)$ exists for $\lambda > \omega$. Then $R(t)$ is a k-regularized resolvent for* (1.1) *if and only if for every $\lambda > \omega$, $(I - \hat{a}(\lambda)A)^{-1}$ exists in $\mathscr{B}(X)$ and*

$$\hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}x = \int_0^\infty e^{-\lambda s}R(s)x\,ds \qquad \text{for all } x \in X.$$

Proof. Assume $R(t)$ is a k-regularized resolvent for (1.1). By assumption, the Laplace transform $H(\lambda) = \hat{R}(\lambda)$ of the k-regularized resolvent exists for $\lambda > \omega$. We claim that $1/\hat{a}(\lambda) \in \rho(A)$ and

$$H(\lambda) = \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}. \tag{3.1}$$

In fact, to compute $H(\lambda)$ we use (R3) and (2.2), obtaining from the convolution theorem for $\lambda > \omega$ the relations

$$H(\lambda)x = \hat{k}(\lambda)x + \hat{a}(\lambda)H(\lambda)Ax, \qquad x \in D(A),$$

and

$$H(\lambda)x = \hat{k}(\lambda)x + A\hat{a}(\lambda)H(\lambda)x, \qquad x \in X.$$

Thus the operators $I - \hat{a}(\lambda)A$ are invertible, and we obtain (3.1).

Conversely, let $\mu, \lambda > \omega$ and $x \in D(A)$. Then $x = (I - \hat{a}(\mu)A)^{-1}y$ for some $y \in X$. Since $(I - \hat{a}(\lambda)A)^{-1}$ and $(I - \hat{a}(\mu)A)^{-1}$ are bounded and commute, and since A is closed,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t}R(t)x\,dt &= \hat{R}(\lambda)(I - \hat{a}(\mu)A)^{-1}y \\
&= \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}(I - \hat{a}(\mu)A)^{-1}y \\
&= (I - \hat{a}(\mu)A)^{-1}\hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}y \\
&= (I - \hat{a}(\mu)A)^{-1}\hat{R}(\lambda)y \\
&= \int_0^\infty e^{-\lambda t}(I - \hat{a}(\mu)A)^{-1}R(t)y\,dt.
\end{aligned}$$

Hence, by uniqueness of the Laplace transform,

$$R(t)x = (I - \hat{a}(\mu)A)^{-1}R(t)(I - \hat{a}(\mu)A)x$$

for almost all $t \geq 0$.

Since $\hat{a}(\mu) \neq 0$ and $R(t)$ is strongly continuous, we conclude that $R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$ for every $t \geq 0$, $x \in D(A)$.

To prove (R3), let $\lambda > \omega$ and $x \in D(A)$. Then, from the convolution theorem we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t} k(t)x\,dt &= \hat{k}(\lambda)x = \hat{R}(\lambda)(I - \hat{a}(\lambda)A)x \\ &= \hat{R}(\lambda) - \hat{R}(\lambda)\hat{a}(\lambda)Ax \\ &= \int_0^\infty e^{-\lambda t}\left[R(t)x - \int_0^t a(t-s)R(s)Ax\,ds\right]dt.\end{aligned}$$

The uniqueness theorem and the strong continuity of $R(t)$ yield the k-regularized resolvent equation. Finally, by Theorem 2.2 we also get $R(0) = k(0)I$ and the proof is complete. ∎

DEFINITION 3.2. A family $\{U(t)\}_{t \geq 0} \subset \mathscr{B}(X)$ is called locally Lipschitz continuous if for $T > 0$ there exists a constant $C_T > 0$ such that $\|U(t) - U(s)\| \leq C_T|t-s|$ for $s, t \in [0, T]$.

The following theorem is a consequence of Proposition 3.1 and the vector-valued version of Widder's theorem given in [1, Theorem 4.1].

THEOREM 3.3. *Let A be a closed and linear operator. Suppose $(I - \hat{a}(\lambda)A)^{-1}$ exists in $\mathscr{B}(X)$ for every $\lambda > \omega$, and $H(\lambda) = \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$ satisfies the estimates*

$$\left\|H^{(n)}(\lambda)\right\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \qquad \lambda > \omega,\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Then (1.1) *admits a locally Lipschitz continuous* $(1 * k)$*-regularized resolvent*.

The next generation theorem is another immediate result. The proof can be carried out as in the case of resolvent families (see [18, Theorem 1.3, p. 43]), and therefore it is omitted.

THEOREM 3.4. *Let A be a closed linear densely defined operator in a Banach space X. Then* (1.1) *admits a k-regularized resolvent $R(t)$ of type (M, ω) if and only if the following conditions hold*:

(H1) $\hat{a}(\lambda) \neq 0$ *and* $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ *for all* $\lambda > \omega$;

(H2) $H(\lambda) = \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$ *satisfies the estimates*

$$\left\|H^{(n)}(\lambda)\right\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \qquad \lambda > \omega,\ n \in \mathbb{N}_0. \tag{3.2}$$

Remark 3.5. In the case where $k(t) \equiv 1$, Theorem 3.4 is well known. In fact, if $a(t) \equiv 1$, then it is just the Hille–Yosida theorem; if $a(t) \equiv t$, then it is the generation theorem for generators of cosine functions due to Sova and Fattorini; for arbitrary $a(t)$, it is the generation theorem due essentially to Da Prato and Iannelli [17]. In the case where $k(t) = t^n/n!$ and $a(t) \equiv 1$, it is the generation theorem for n-times integrated semigroups [10]; if $k(t) = t^n/n!$ and $a(t)$ is arbitrary, it corresponds to the generation theorem for integrated solutions of Volterra equations due to Arendt and Kellermann [2].

In general, (H2) is difficult to check; however, we will show that there are important special classes of operators A and kernels $a(t)$ such that this is possible.

We recall that a C^∞-function f: $(0, \infty) \to \mathbb{R}$ is called *completely monotonic* if $(-1)^n f^{(n)}(\lambda) \geq 0$ for all $\lambda > 0$, $n \in \mathbb{N}_0$, and a *Bernstein function* if $f(\lambda) \geq 0$ and $f'(\lambda)$ is completely monotonic.

A function $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is called *completely positive* if the solutions $s_\epsilon(t)$ and $r_\epsilon(t)$ of the convolution equations

$$s_\epsilon + \epsilon a * s_\epsilon = 1 \qquad \text{and} \qquad r_\epsilon = \epsilon a * r_\epsilon = a \tag{3.3}$$

are both nonnegative for each $\epsilon > 0$.

Remark 3.6. Assume $a(t)$ is Laplace transformable and $\hat{a}(\lambda) \neq 0$ for all $\lambda > 0$. Then $a(t)$ is completely positive if and only if $1/\lambda\hat{a}(\lambda)$ is completely monotonic and $1/\hat{a}(\lambda)$ is a Bernstein function. For other equivalences, see also [17, 18].

Denote by $a^{*m} = a * a * \cdots * a$ the m-times convolution of the kernel $a(t)$.

The next is the main result of this section.

THEOREM 3.7. *Let A be the generator of an m-times integrated semigroup of type (M, ω_0), $\omega_0 \geq 0$, and assume $a(t)$ is Laplace transformable and completely positive. Then* (1.1) *admits a locally Lipschitz continuous $(t * a^{*m})$-regularized resolvent. If, additionally A is densely defined, then* (1.1) *admits an exponentially bounded $(1 * a^{*m})$-regularized resolvent.*

Proof. The proof is in the same spirit as Theorem 5 in [17]; thus we only sketch it. Let $S(t)$ be the m-times integrated semigroup generated by A. We have the identity

$$(z - A)^{-1} = z^m \int_0^\infty S(\tau) e^{-z\tau}\, d\tau \tag{3.4}$$

for all $z > \omega_0$. Since $\hat{a}(\lambda) > 0$ for λ sufficiently large, $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$ implies $1/\hat{a}(\lambda) > \omega_0$, say for $\lambda > \omega$; substituting $z = 1/\hat{a}(\lambda)$ in (3.4) then leads to

$$H(\lambda) := \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1} = \int_0^\infty S(\tau)h(\lambda, \tau)\, d\tau, \tag{3.5}$$

for all $\lambda > \omega$, where $k(t) = \int_0^t a^{*m}(s)\, ds$ and

$$h(\lambda, \tau) = \frac{1}{\lambda \hat{a}(\lambda)} e^{-\tau/\hat{a}(\lambda)}. \tag{3.6}$$

Since $a(t)$ is completely positive, we obtain that $h(\lambda, \tau)$ is completely monotonic with respect to $\lambda > 0$ for each fixed $\tau \geq 0$ (see, e.g., [18, Proposition 4.5]).

Let L_λ^n denote the Widder operators $L_\lambda^n = (-1)^n (d/d\lambda)^n / n!$. By representation (3.5) we obtain

$$\begin{aligned} \| L_\lambda^n(\lambda) \| &\leq \int_0^\infty L_\lambda^n h(\lambda, \tau) \| S(\tau) \| \, d\tau \leq M \int_0^\infty L_\lambda^n h(\lambda, \tau) e^{\omega\tau}\, d\tau \\ &= M L_\lambda^n \int_0^\infty h(\lambda, \tau) e^{\omega\tau}\, d\tau = M L_\lambda^n \left[\frac{1}{\lambda(1 - \omega \hat{a}(\lambda))} \right]. \end{aligned}$$

Since $a(t)$ is a completely positive function, we can prove that there is some function $l \in BV_{\mathrm{loc}}(\mathbb{R}_+)$ such that $0 \leq l(t) \leq c e^{\omega_1 t}$ for all $t \geq 0$, which satisfies

$$\frac{1}{\lambda(1 - \omega \hat{a}(\lambda))} = \hat{l}(\lambda),$$

for all $\lambda > \omega_1$ (see [18, p. 102]). Hence

$$0 \leq L_\lambda^n \big[\hat{l}(\lambda) \big] \leq \frac{c}{(\lambda - \omega_1)^{n+1}}, \qquad \lambda > \omega_1, n \in \mathbb{N}_0.$$

Thus,

$$\| L_\lambda^n H(\lambda) \| \leq \frac{cM}{(\lambda - \omega_1)^{n+1}}, \qquad \lambda > \omega_1, n \in \mathbb{N}_0.$$

The generation Theorem 3.3 then applies and therefore Eq. (1.1) admits a locally Lipschitz continuous $(1 * 1 * a^{*m})$-regularized resolvent. Finally, applying the generation Theorem 3.4 instead of Theorem 3.3, we obtain the existence of an exponentially bounded $(1 * a^{*m})$-regularized resolvent in the case that the domain of A is densely defined. ∎

The following corollary recovers a result of Prüss [17, Theorem 5].

COROLLARY 3.8. *Let A be the generator of a C_0-semigroup and let $a(t)$ be completely positive. Then* (1.1) *admits a resolvent.*

Remark 3.9. In the case $m = 1$, Oka [16] has proved, using completely different methods, the existence of a locally Lipschitz continuous t-regularized resolvent for (1.1). However, Oka's theorem requires $a \in AC_{\text{loc}}(\mathbb{R}_+)$, $a' \in BV_{\text{loc}}(\mathbb{R}_+)$, and $a(0) = 1$.

EXAMPLE 3.10. Consider the equation

$$\frac{\partial u}{\partial t}(t, x) = Au(t, x) - \alpha \int_0^t e^{-\alpha(t-s)} Au(s, x)\, ds,$$

$$\alpha \geq 0,\ 0 \leq t \leq T,\ x \in \mathbb{R}, \quad (3.7)$$

$$u(0, x) = u_0(x),$$

where the operator $A := a(\partial^3/\partial x^3) + b(\partial/\partial x)$ $(a, b \in \mathbb{R}\backslash\{0\})$ is defined in $L^p(\mathbb{R})$ with maximal domain. Note that, for $\alpha = 0$, Eq. (3.8) is just the linearized Korteweg–de Vries equation

$$\frac{\partial u}{\partial t}(t, x) = a\frac{\partial^3 u}{\partial x^3}(t, x) + b\frac{\partial u}{\partial x}(t, x).$$

To apply Theorem 3.7 observe that (3.7) is equivalent to (1.1) with $f(t, x) = u_0(x)$ and $a(t) = e^{-\alpha t}$; where $\frac{1}{\hat{a}(\lambda)} = \alpha + \lambda$ is a Bernstein function and $\frac{1}{\lambda \hat{a}(\lambda)} = \frac{\alpha}{\lambda} + 1$ is completely monotonic.

From [8, p. 15] we know that A is the generator of a once integrated semigroup on $L^p(\mathbb{R})$ for $p > 1$. We conclude that there is a locally Lipschitz continuous k-regularized resolvent for (3.7) with $k(t) = \int_0^t (t - s)a(s)\, ds$.

Define the operator $Bv(t) = \alpha v''(t) + v'''(t)$ on $C_0^3([0, T], L^p(\mathbb{R}))$. It is not difficult to see that the convolution transform $Ku(t) = (k * u)(t)$ verifies $KBv(t) = v(t)$ and thus $\text{Ran}(K) = C_0^3([0, T], L^p(\mathbb{R}))$ by Remark 2.8. Applying Theorem 2.7 and Remark 2.10, we conclude that there exists a unique solution $u \in C([0, T]; L^p(\mathbb{R}))$ of (3.7) for all initial values $u_0 \in W^{4,p}(\mathbb{R})$ $(1 < p < \infty)$.

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